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Solution In Degenerate Ordinary Systems Of Differential Equations By The Differential Sweep Method

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Abstract
In this article, we consider a system of degenerate ordinary differential equations. A calculation method using simple factorization is proposed. The existence of a solution to the boundary value problem is proved, an algorithm for friction of the problem is constructed, and a uniform estimate for the solution is obtained using the maximum principle method.

Keywords: continuous functions, boundary value problem, factorization, maximum principle.

Introduction

Some problems of gas and liquid filtration in a three-layer reservoir are reduced to solving boundary value problems for systems of degenerate differential equations [1]. Consider a boundary value problem for a system of degenerate differential equations of the form:

$$\begin{cases} \dfrac{1}{m(x)}\dfrac{d}{dx}\left(K(x)\dfrac{du}{dx}\right) = a(x)u + b(x) + \sum\limits_{i=1}^{2} A_i(x)K_i(z)\left.\dfrac{\partial u_1}{\partial z}\right|_{z=1} \\ \dfrac{1}{m_i(z)}\dfrac{\partial}{\partial z}\left(K_i(z)\dfrac{\partial u_i}{\partial z}\right) = a_i(z)u_i(x,z) + b_i(x,z) \end{cases} \qquad (1)$$

In area $\Omega = \{0 < x < 1, 0 < z < 1\}$

Here $K(x), m(x), A_i(x), K_i(z), m_i(z), a_i(z)$ - given functions on the segment $[0,1]$, $b_1(x,z)$ are given functions in $\overline{\Omega}$, and $K(x), m(x), a_i(z), a(x) > K_0(0)$ $K_i(0) = 0$, $K_i(z)$ and $m_i(z)$ are positive for $z > 0$.

The boundary conditions have the form

$$\begin{cases} \gamma_0 K(x)\dfrac{du}{dx}\Big|_{x=0} - \alpha_0 u\big|_{x=0} - \beta_0 = 0 \\ \gamma_1 K(x)\dfrac{du}{dx}\Big|_{x=1} - \alpha_1 u\big|_{x=1} - \beta_1 = 0 \\ \gamma_{1i} K_i(z)\dfrac{du_i}{dz}\Big|_{z=0} - \alpha_{1i} u_i\big|_{z=0} - \beta_{1i} = 0 \\ if \quad g_i(0) < +\infty \\ u(x) = u_i(x,1), \ \ i = 1,2 \end{cases} \qquad (2)$$

If $g_i(0) < +\infty$, and $\sigma_i(0) < +\infty$ then the condition for $z = 0$ is replaced by the condition $\left| u_i(x,z)\big|_{z=0} \right| < +\infty, \ \ i = 1,2$.

Here

$\gamma_k + \alpha_k \neq 0 \ \ k = 0,1; \ \ \gamma_{1i} + \alpha_{1i} \neq 0, \ \ i = 1,2$

$$g_i(z) = \int_0^1 \frac{d\xi}{K_i(\xi)}, \quad \sigma_i(z) = \int \frac{\int_0^1 m_i(\xi)d\xi}{K_i(\eta)} d\eta$$

To prove the existence of a solution to the considered boundary value problem, we need the following:

Lemma 1.1. Let $K_i(z), m_i(z), a_i(z) \in [0,1]$, $a_i(z) \geq a_{i0} > 0, K_i(0) = 0$ and $m_i(z)$ be positive . Then $[0,1]$ there is a unique continuous solution of the equation on the interval $\frac{1}{m_i(z)} \frac{d}{dz}(K_i(z)\frac{dV_i}{dz}) = a_i(z)V_i(z), \ \ i = 1,2$ satisfying one of the initial conditions: $V_i\big|_{z=0} = h_1, K_i(z)\frac{dV_i}{dz} = \mu_i$. If $g_i(0) < +\infty$ (4). If $g_i(0) < +\infty$, a $\sigma_1(0) < +\infty$ then $V_i\big|_{z=0} = h_1, \ \ i = 1,2$. Here are $\mu_i, h_i \ \ (i = 1,2)$ some constants.

Proof. Consider the case $g_i(0) < +\infty$

It is easy to see that problems (3) , (4) are equivalent to the system of integral equations

$$V_i(z) = h_1 + \mu_1 \int_0^z \frac{d\xi}{K_i(\xi)} + \int_0^z \frac{\int_0^\xi m_i(\eta)a_i(\eta)V_i(\eta)d\eta}{K_i(\xi)} d\xi$$

Using the contraction mapping principle, let us show the unique solvability of the system of integral equations (5) in the class of two-component vector functions.

$$\bar{\sigma}_i(z) = \int_0^z \frac{\int_0^\xi m_i(\eta)a_i(\eta)V_i(\eta)d\eta}{K_i(\xi)} d\xi, \qquad i = 1,2$$

It is easy to see that the conditions of the lemma ensure the continuity and monotonicity of the function $\bar{\sigma}_i(z)$. Obviously, $\bar{\sigma}_i(0) = 0$ therefore, it is possible to choose $\delta \in (0,1)$ such that $\bar{\sigma}_i(\delta) < 1$. Let us denote $\bar{\Omega}_\delta = [0, \delta]$ and introduce into consideration the complete space of two-component continuous vector functions $C\bar{\Omega}_\delta$. In space, $C\bar{\Omega}_\delta$ consider the image

$$AV_i(z) = h_1 + \mu_1 \int_0^z \frac{d\xi}{K_i(\xi)} + \int_0^z \frac{\int_0^\xi m_i(\eta) a_i(\eta) V_i(\eta) d\eta}{K_i(\xi)} d\xi$$

Let us show that A translates $C\bar{\Omega}_\delta$ into itself. Let $\{V_1(z), V_2(z)\} \in C\bar{\Omega}_\delta$ then $z_1, z_2 \in C\bar{\Omega}_\delta$ the estimate

$$\left| AV_i(z_2) - AV_i(z_i) \right| \le \mu_i \left| \int_{z_1}^{z_2} \frac{d\xi}{K_i(\xi)} \right| + \max_i \left| V_i(z_2) - V_i(z_1) \right| \cdot \left| \bar{\sigma}_i(z_2) - \bar{\sigma}_i(z_1) \right|$$

Due to the continuity of the functions $\bar{\sigma}_i$ and the convergence of the integrals $\int_0^\delta \frac{dz}{K_i(z)}$ from $z_1 \to z_2$ follows $AV_i(z_1) \to AV_i(z_2)$ i.e. $\{AV_1(z), AV_2(z)\} \in C(\bar{\Omega}_\delta)$

Similarly, we have

$$\left| AV_i(z) - A\tilde{V}_i(z) \right| \le \max_{z \in \Omega_\delta} \left| V_i(z) - \tilde{V}_i(z) \right| \bar{\sigma}_i(\delta)$$

From there, by virtue of inequalities $\bar{\sigma}_i(\delta) < 1$, the contraction of the mapping A follows. From here, by virtue of the Banach fixed point theorem, it follows that the system of equations (5) has a unique solution in the space $C\bar{\Omega}_\delta$. Due to the linearity of the equations and the fact that the coefficients of problem (3), (4) do not have $[0,1]$ singularities on the segment, this solution can be extended continuously to the segment.

The case is considered similarly $g_i(0) < +\infty,\ \sigma_i(0) < +\infty$. In this case, the relation

$$\lim_{z \to 0} K_i(z) \frac{dV_i}{dz} = 0$$

The validity of which is easy to obtain from the requirement that the solution be bounded $V_i(z)$.

Lemma 2. Let the conditions of Lemma 1 be satisfied, moreover, $a(x),\ b(x),\ A_i(x),\ K(x), m(x) \in C[0,1],\ b_1(x,z) \in C(\bar{\Omega})$. Then there is a unique solution of the system of equations (1) that satisfies conditions (2) and is continuous together with the derivatives $\frac{d}{dx} K(x) \frac{du}{dx}) b(0,1)$ And $\frac{1}{m_i(z)} \frac{\delta}{\delta_z} (K_i(z) \frac{\delta u_i}{\delta z})$

In area Ω this solution can be constructed using the differential sweep method.

Proof. Let's build functions

$$\alpha_1(z) = \frac{1}{V_i(z)} \left(\frac{\alpha_{1i}}{\gamma_{1i}} + \int_0^z m_i(\xi) a_i(\xi) V_i(\xi) d\xi \right. \tag{6}$$

$$\beta_1(x,z) = \frac{1}{V_i(z)} \left(\frac{\beta_{1i}}{\gamma_{1i}} + \int_0^z m_i(\xi) b_i(x,\xi) V_i(\xi) d\xi \right), \tag{7}$$

Where is $V_i(z)$ the solution of problem (3), (4), where

$\mu_i = \dfrac{\alpha_{1i}}{\gamma_{1i}}, \quad h_1 = 1, \; i = 1,2$

From (3), (4) we easily obtain

$$K_i(z)\frac{dV_i}{dz} = \frac{\alpha_{1i}}{\gamma_{1i}} + \int_0^z m_i(\xi)a_i(\xi)V_i(\xi)d\xi \qquad (8)$$

Comparing (6) and (8), we obtain the relation

$$\alpha_i(z) = \frac{K_i(z)\dfrac{dV_i}{dz}}{V_i(z)} \qquad (9)$$

Ras c look functions

$$\alpha(x) = \frac{1}{V(x)}\left[\frac{\alpha_0}{\gamma_0} + \int_0^x\left[\alpha(\xi) + \sum_{i=1}^{2} A_i(\xi)\alpha_i(1)\right]\right]m(\xi)V(\xi)d\xi \qquad (10)$$

$$\beta(x) = \frac{1}{V(x)}\left[\frac{\alpha_0}{\gamma_0} + \int_0^x\left[b(\xi) + \sum_{i=1}^{2} A_i(\xi)\alpha - \beta_i(1)\right]\right]m(\xi)V(\xi)d\xi \qquad (11)$$

Where $V(x)$ solution of the Cauchy problem

$$\frac{1}{m(x)}\frac{d}{dx}(K(x)\frac{dV}{dx}) = \left[a(x) + \sum_{i=1}^{2} A_i(x)\alpha_i(1)\right]V(x)$$

$$V(0) = 1, \quad K(x)\frac{du}{dx}\bigg|_{x=0} = \frac{\alpha_0}{\gamma_0} \qquad (12)$$

It is easy to prove that problem (12) has a continuous monotone unique solution, since $\alpha_1(1) > 0$. It is easy from (10) , (12) to obtain the relation

$$\alpha(x) = \frac{K(x)\dfrac{dV}{dx}}{V(x)} \qquad (13)$$

Let us now construct the functions $u(x)$ And $u_i(x,z)$ using formulas

$$u(x) = \frac{V(x)}{V(1)}(u(1) - \int_x^1 \frac{\beta(\xi)V(1)}{K(\xi)V(\xi)}d\xi) \; (14)$$

$$u_i(x,z) = \frac{V_i(x)}{V_i(1)}(u(x) - \int_x^1 \frac{\beta_i(x,\xi)V_i(1)}{K_i(\xi)V_i(\xi)}d\xi) \; (15)$$

Where $u(x,1) = u(x)$

$$u(1) = -\frac{\beta_i + \gamma_i\beta(1)}{\alpha_i + \gamma_i\alpha(1)} \; (16)$$

Let us now show the continuity $u(x,1)$ and $u(x)$

From Lemma 1 it follows that $V_i(z)$ does not decrease on the interval $[0,1]$ and therefore , then from (6), (7) it follows that the functions $V_i(z) \geq 1$ and $\beta_1(x,z)$ $\beta_i(x,z)$ are continuous $\alpha_1(z)$ $\alpha_i(z)$ for $z > 0$, from (14) it follows that $u(x)$ is continuous for $x \in [0,1]$. Then it follows from (15) that is $u_i(x,z)$ continuous in the domain $(0,1] \times (0,1]$.

To $u_i(x,z)$ be continuous in the domain Ω, it suffices to prove the uniform convergence of the integral

$$\int_0^1 \frac{\beta_i(x,\xi)}{K_i(\xi)V_i(\xi)}d\xi \quad (17)$$

In the case under consideration, the uniform convergence of the integral (17) is obvious (because $g_i(0) < +\infty$), hence $u_i(x,z)$ it is continuous in the region $\overline{\Omega}$

Applying the maximum principle, we obtain the uniform estimate

$$\{u(x); u_i(x,z)\} \le \max\left\{\left|\frac{a(x)}{b(x)}\right|; \max_2\left|\frac{\alpha_i(z)}{b_i(x,z)}\right|\right\}$$

Numerical solution of nonlinear filtering problems using the method of straight lines in a variable t the original problem is replaced by a differential-difference problem. To solve the differential-difference problem, an iteration method is proposed, as a result we get a system of ordinary differential equations. It was possible to obtain modernized versions of the differential sweep method in relation to the problems under consideration.

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